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EE 364 - Spring 2026

Week 13

Supplemental Notes

Topics

1. Conditioning
   - total expectation
   - total variance
   - total covariance
2. "SIT" technique

Conditional Estimation

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \text{for } P(A) > 0$$

Conditional pdf

$$\underbrace{f_{Y|X}(y|x)}_{\text{conditional}} = \frac{f_{XY}(x,y)}{f_X(x)}$$

joint

marginal

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

Total Probability : $P(B) = P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c)$

$$P(E) = \sum_k P(E|H_k) \cdot P(H_k)$$ if $\{H_k\}$ partition Ω

Thrm: $$P(X \in A) = \int_{y=-\infty}^{y=\infty} P(X \in A | Y = y) \cdot f_Y(y)\, dy$$

$$= E_Y[P(X \in A | \overset{\text{r.v.}}{Y})]$$

Prf: $$P(X \in A) = P\big((X,Y) \in A \times \mathbb{R}\big)$$ if $A \subset \mathbb{R}$.

$$= \iint_{A \times \mathbb{R}} f_{XY}(x,y)\, dy\, dx$$

$$\overset{\text{(Fubini)}}{=} \int_A \int_{y=-\infty}^{y=\infty} f_{XY}(x,y) \cdot dy\, dx$$

since $f(x|y) = \frac{f(x,y)}{f(y)}$

$$= \int_{y=-\infty}^{y=\infty} \left(\int_A f_{X|Y}(x|y)\, dx \right) f_Y(y)\, dy.$$

$$= \int_{y=-\infty}^{y=\infty} P[X \in A | Y = y]\, f_Y(y)\, dy.$$

QED

Similarly:

Thrm: $$P[(X,Y) \in B] = \int_{x=-\infty}^{x=\infty} P[(X,Y) \in B | X = x] \cdot f_X(x)\, dx.$$

★ "SIT" technique for solving problems

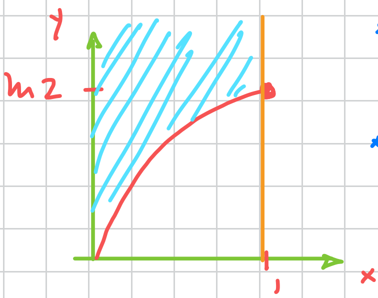
(1) T: use Total probability (or expectation/variance/etc.)

(2) S: Substitute x for r.v. X if $X=x$.

(3) I: Use Independence of X and Y (if appropriate)

Ex: Independent X and Y: $X \sim U[0,1]$, $Y \sim \text{Exp}(1)$

find $P(Y > \ln(1+X))$



$$\int_{x=0}^{x=1}\int_{y=\ln(1+x)}^{y=\infty} 1-e^{-x}\, dx.$$

(use SIT)

$$P(Y > \ln(1+X)) = P((X,Y) \in B)$$

$$\overset{\text{Total prob}}{=} \int_{x=-\infty}^{x=\infty} P(Y > \ln(1+X) \mid X=x)\, f_X(x)\, dx$$

$$\overset{X\sim U[0,1]}{=} \int_0^1 P(Y > \ln(1+X) \mid X=x)\, dx$$

$$\overset{\text{Sub}}{=} \int_0^1 P(Y > \ln(1+x) \mid X=x)\, dx$$

$$\overset{\text{Ind}}{=} \int_0^1 P(Y > \ln(1+x))\, dx$$

$$\overset{Y\sim \text{Exp}(1)}{=} \int_0^1 e^{-\ln(1+x)}\, dx$$

$$= \int_0^1 e^{\ln\left(\frac{1}{1+x}\right)}\, dx$$

$$= \int_0^1 \frac{dx}{1+x}$$

put $u = 1+x$, $du = dx$ $\therefore$ $x=0 \longrightarrow u=1$, $x=1 \longrightarrow u=2$

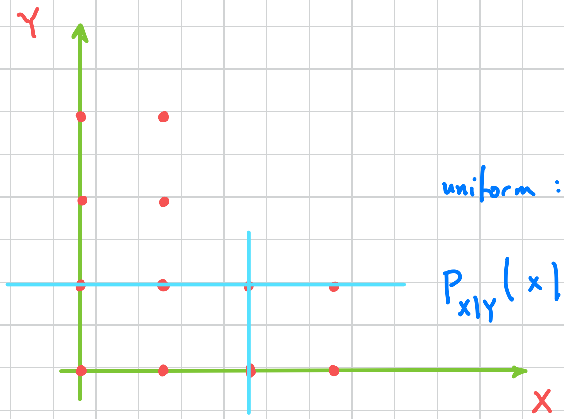
$$\therefore \overset{u=1+x}{=} \int_1^2 \frac{du}{u} = \ln u \Big|_1^2 = \ln 2 - \cancel{\ln 1} = \ln 2 \approx 0.693$$

Defn: Conditional Expectation r.v. $E[Y|X]$:

$\forall x$: $E[Y|X=x] = \begin{cases} \int_{y=-\infty}^{y=\infty} y \cdot f_{Y|X}(y|x)\, dy & \text{continuous} \\ \sum_y y \cdot p_{Y|X}(y|x) & \text{discrete} \end{cases}$

☆☆ Note: $E[Y|X=x] = g(x)$ ← constant $\therefore g(X) = E[Y|X]$. ← random variable

Ex:



uniform: $P(\text{point}) = \frac{1}{12}$

$P_{X|Y}(x|y=1)$

$P_{Y|X}(y|x=2)$

unconditional expectation $E[Y] = 0 \cdot \frac{4}{12} + 1 \cdot \frac{4}{12} + 2 \cdot \frac{2}{12} + 3 \cdot \frac{2}{12}$

$= \underbrace{1\tfrac{1}{6}}_{\text{constant.}}$

$E[Y|X=0] = 1.5$

$E[Y|X=2] = 0.5$

$E[X|Y=0] = 1.5$

$E[Y|X] = \begin{cases} 1.5 & p = \frac{1}{3} & (x=0) \\ 1.5 & p = \frac{1}{3} & (x=1) \\ 0.5 & p = \frac{1}{6} & (x=2) \\ 0.5 & p = \frac{1}{6} & (x=3) \end{cases}$

probability measure (i.e. random variable)

sample space

☆☆☆ $E[Y|X]$ is a random variable since expected value Y depends on random variable X $E[Y|X] \sim g(X)$

constant

Q: expected value of $E[Y|X]$?

A: $E_X[\underbrace{E[Y|X]}_{\text{r.v.}}] = \frac{1}{3} \cdot \frac{3}{2} + \frac{1}{3} \cdot \frac{3}{2} + \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{2} = 1\frac{1}{6}$ ← $= E_Y[Y]$

Thrm: (Total Expectation)

$$E_X[X] = E_Y[E[X|Y]] \quad \text{(if exists)}$$

Prf:
$$E_Y[E(X|Y)] \overset{\text{cont.}}{=} \int_{y=-\infty}^{y=\infty} E[Y|X=x] \cdot f_Y(y)\, dy \quad \text{(assume exists)}$$
$$= \int_{y=-\infty}^{y=\infty} \int_{x=-\infty}^{x=\infty} x \cdot f(x|y) \cdot f_Y(y)\, dy$$
$$\overset{\text{Fubini}}{=} \int_{x=-\infty}^{x=\infty} x \left[\int_{y=-\infty}^{y=\infty} f_{XY}(x,y)\, dy \right] dx \quad \text{since } f(x|y) = \frac{f(x,y)}{f(y)}$$
$$= \int_{x=-\infty}^{x=\infty} x \cdot f_X(x)\, dx$$
$$= E_X[X]$$

QED

Ex: Bernoulli, H: \$10, T: \$2

$$E_X[X] = E_Y[E[X|Y]]$$
$$\therefore E_X[X] = P(Y=0) \cdot E[X|Y=0] + P(Y=1) \cdot E[X|Y=1]$$

Similarly:

(1) $$E_{XY}[g(X,Y)] = \int_{x=-\infty}^{x=\infty} E[g(X,Y) | X=x] \cdot f_X(x)\, dx$$

(2) $$E_{XYZ}[g(X,Y,Z)] =$$
$$\int_{z=-\infty}^{z=\infty} \int_{y=-\infty}^{y=\infty} E[g(X,Y,Z) | Y=y, Z=z] \cdot f_{YZ}(y,z)\, dy\, dz$$

Conditioning

Total Expectation

$$E_X[X] = E_Y\big[E[X|Y]\big].$$

Total Variance

$$V_X[X] = E_Y\big[V[X|Y]\big] + V_Y\big[E[X|Y]\big].$$

Total Covariance

$$\text{Cov}(X,Y) = E_Z\big[\text{Cov}(X,Y|Z)\big] + \text{Cov}_Z\big(E[X|Z],\, E[Y|Z]\big)$$

Ex: Dependent X and Y: $\quad X \sim \text{Exp}(1), \quad Y_{|X=x} \sim N(0, \underset{x>0}{x^2})$

Find $E[Y^2X^4]$

use TS of "SIT"

$$\therefore\; E[Y^2X^4] \overset{\text{total exp.}}{=} \underbrace{\int_{x=-\infty}^{x=\infty} E[Y^2X^4 \mid X=x]\, f_X(x)\,dx}_{E_X[E[Y^2X^4|X]]}.$$

$$\overset{X\sim \text{Exp}(1)}{=} \int_0^\infty E[Y^2X^4 \mid X=x]\, e^{-x}\,dx$$

$$\overset{\text{Sub}}{=} \int_0^\infty E[Y^2x^4 \mid X=x]\, e^{-x}\,dx$$

$$= \int_0^\infty x^4 \cdot E[Y^2 \mid X=x]\, e^{-x}\,dx \qquad \text{(Proposition 1, below)}$$

$$= \int_0^\infty x^4 x^2 e^{-x}\,dx \qquad \text{since } Y_{|X=x} \sim N(0, x^2)$$

$$= \int_0^\infty x^6 e^{-x}\,dx$$

$$= E_X[X^6]$$

$$= \underline{6!} = 720 \qquad \text{Since } X \sim \text{Exp}(1)$$

Ex: Suppose $X, Y \sim \exp(1)$. Put $Z = X/Y$. find $f_Z(z)$.

Suppose $Y = y$. (i.e. Y is constant)

then $Z = X/y$ is a scaled version of X (scaled by constant $1/y$)

$\therefore f_Z(z|y) = |y| \cdot f_X(yz|y)$ see earlier example $Y = aX+b$ $(a \neq 0)$ $\longrightarrow$ $Y = aX$

$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y}{a}\right)$

$$\therefore f_Z(z) = \int_{-\infty}^{\infty} |y| \cdot \underbrace{f_X(y \cdot z|y)}_{\text{conditional}} \cdot \underbrace{f_Y(y)}_{\text{marginal}} \, dy$$

$$\overset{\text{indep.}}{=} \int_{-\infty}^{\infty} |y| \cdot f_X(yz) \cdot f_Y(y) \, dy \overset{\exp}{=} \int_0^{\infty} y e^{-yz} e^{-y} \, dy = \int_0^{\infty} y e^{-\overset{a}{(1+z)}y} \, dy$$

Let $u = y$, $du = dy$; $dv = e^{-ay} dy$, $v = e^{-ay}/-a$

$$= \cancel{\frac{ye^{-ay}}{-a}}^{=0} \Bigg|_{y=0}^{y=\infty} + \int_0^{\infty} \frac{e^{-ay}}{+a} \, dy$$

$$= -\frac{e^{-ay}}{a^2} \Bigg|_{y=0}^{y=\infty} = \frac{1}{(1+z)^2} \quad \text{for } z \geq 0$$

Proposition 1: $E[g(X)Y|X] = g(X) \cdot E[Y|X]$.

Prf: $\forall x$:

$$E[g(X)Y|X=x] = \int_{y=-\infty}^{y=\infty} g(x) \cdot y \cdot f(y|x) \, dy$$

$$= g(x) \cdot \int_{y=-\infty}^{y=\infty} y \cdot f(y|x) \, dy$$

$$= g(x) \cdot E[Y|X=x].$$

$$\therefore E[g(X) \cdot Y|X] = g(X) \cdot E[Y|X].$$

Defn: Conditional Variance $V[Y|X] = E[(Y - E[Y|X])^2 | X]$ (if exists)

Fact: If X and Y are Jointly Gaussian (JG)

$$f_{XY}(x,y) = \frac{\exp\left[-\dfrac{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho_{XY}\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)}{2(1-\rho_{XY})^2}\right]}{2\pi\,\sigma_x\sigma_Y\sqrt{1-\rho_{XY}}}$$

then:

(1) $E[Y|X] = \mu_Y + \rho_{XY}\dfrac{\sigma_Y}{\sigma_x}(X-\mu_x)$

(2) $V[Y|X] = \sigma_Y^2(1-\rho_{XY}^2)$

$\therefore\ \sigma_{Y|X}^2 < \sigma_Y^2$ if $\rho_{XY} = \dfrac{\sigma_{XY}}{\sigma_X\sigma_Y} \neq 0.$

Ex: Q1: roll 6 dice (iid)

$$E\left[\sum_{k=1}^{6} X_k\right] = \sum_{k=1}^{6} E[X_k] = 6\,E[X] = 6 \cdot 3.5 = 21$$

Q2: roll 1 die, X = outcome. $\underbrace{\text{Roll } X \text{ dice}}_{\text{roll random \# of dice}}$ $\quad E\left[\sum_{k=1}^{\overset{\text{random}}{N}} X_k\right] = ?$

Proposition 2: $V[Y|X] = E[Y^2|X] - E^2[Y|X]$

Prf:
$$V[Y|X] = E[(Y - E[Y|X])^2 | X]$$

$$\overset{\text{Lin}}{=} E[\,Y^2 + E^2[Y|X] - 2 \cdot Y\, E[Y|X] \,|\, X\,]$$

$$\overset{\text{Lin}}{=} E[Y^2|X] + E[\overbrace{E^2[Y|X]}^{g(x)} | X]$$

$$- 2 \cdot E[Y\,\overbrace{E[Y|X]}^{h(x)} | X]$$

$$\overset{\text{prop 1}}{=} E[Y^2|X] + E^2[Y|X] \cdot E[1|X]$$

$$- 2 \cdot E[Y|X] \cdot E[Y|X]$$

$$= E[Y^2|X] + E^2[Y|X] - 2 \cdot E^2[Y|X]$$

$$= E[Y^2|X] - E^2[Y|X]$$

QED

Thrm: (Total Variance)

$$V_X[X] = E_Y[V[X|Y]] + V_Y[E[X|Y]].$$ "ev-ve"

Prf:
$$E_Y[V[X|Y]] + V_Y[E[X|Y]]$$

$$= E_Y[V[X|Y]] + E_Y[E^2[X|Y]] - (E_Y[E[X|Y]])^2$$

$$\overset{\text{total exp}}{=} E_Y[V[X|Y]] + E_Y[E^2[X|Y]] - E_X^2[X].$$

$$\overset{\text{prop 2}}{=} E_Y[E[X^2|Y] - E^2[X|Y]] + E_Y[E^2[X|Y]] - E_X^2[X]$$

$$= E_Y[E[X^2|Y]] - \cancel{E_Y[E^2[X|Y]]}$$

$$+ \cancel{E_Y[E^2[X|Y]]} - E_X^2[X]$$

$$\overset{\text{total exp}}{=} E_X[X^2] - E_X^2[X]$$

$$= V_X[X]$$

QED

★ **Thrm:** (Doubly Random Sum) If iid $X_1, \dots, X_n$ $\forall n$ and $\sigma_X^2 < \infty$ ($\therefore \mu_X < \infty$).

- Positive $\mathbb{Z}^+$ r.v. N
- N is independent of $X_1, \dots, X_n$ $\forall n$.

$$(1)\quad E\left[\sum_{k=1}^{N} X_k\right] = E_N[N]\cdot \mu_X$$

$$(2)\quad V\left[\sum_{k=1}^{N} X_k\right] = E_N[N]\,\sigma_X^2 + V_N[N]\,\mu_X^2$$

Prf:

(SIT technique)

$$(1)\quad E\left[\sum_{k=1}^{N} X_k\right] \overset{\text{total exp}}{=} E_N\left[E\left[\sum_{k=1}^{N} X_k \,\middle|\, N\right]\right].$$

$$= \sum_{n=1}^{\infty} p_N(n)\cdot E\left[\sum_{k=1}^{N} X_k \,\middle|\, N=n\right]. \quad \text{since } N>0$$

$$\overset{\text{sub}}{=} \sum_{n=1}^{\infty} p_N(n)\, E\left[\sum_{k=1}^{n} X_k \,\middle|\, N=n\right]$$

$$\overset{\text{ind}}{=} \sum_{n=1}^{\infty} p_N(n)\cdot E\left[\sum_{k=1}^{n} X_k\right]$$

$$\overset{\text{lin}}{=} \sum_{n=1}^{\infty} p_N(n)\cdot \sum_{k=1}^{n} E[X_k].$$

$$\overset{\text{iid}}{=} \sum_{n=1}^{\infty} p_N(n)\,(n\cdot\mu_X)$$

$$= \mu_X \cdot \sum_{n=1}^{\infty} n\cdot p_N(n)$$

$$= \mu_X\, E_N[N] = E_N[N]\cdot \mu_X$$

$$
\begin{aligned}
(2)\quad V\left[\sum_{k=1}^{N} X_k\right] &\overset{\text{total var}}{=} E_N\left[V\left[\sum_{k=1}^{N} X_k \,\middle|\, N\right]\right] + V_N\left[E\left[\sum_{k=1}^{N} X_k \,\middle|\, N\right]\right] \\
&= \sum_{n=1}^{\infty} p_N(n)\cdot V\left[\sum_{k=1}^{N} X_k \,\middle|\, N=n\right] + V_N\left[E\left[\sum_{k=1}^{N} X_k \,\middle|\, N\right]\right] \\
&\overset{\text{sub}}{=} \sum_{n=1}^{\infty} p_N(n)\cdot V\left[\sum_{k=1}^{n} X_k \,\middle|\, N=n\right] + V_N\left[E\left[\sum_{k=1}^{N} X_k \,\middle|\, N\right]\right] \\
&\overset{\text{ind}}{=} \sum_{n=1}^{\infty} p_N(n)\cdot V\left[\sum_{k=1}^{n} X_k\right] + V_N\left[E\left[\sum_{k=1}^{N} X_k \,\middle|\, N\right]\right] \\
&\overset{\text{iid}}{=} \sigma_x^2 \sum_{n=1}^{\infty} n\cdot p_N(n) + V_N\left[E\left[\sum_{k=1}^{N} X_k \,\middle|\, N\right]\right] \\
&= E_N[N]\,\sigma_x^2 + V_N\left[E\left[\sum_{k=1}^{N} X_k \,\middle|\, N\right]\right] \\
&= E_N[N]\,\sigma_x^2 + E_N\left[\left(E\left[\sum_{k=1}^{n} X_k \,\middle|\, N=n\right]\right)^2\right] \\
&\qquad - \left(E_N\left[E\left[\sum_{k=1}^{N} X_k \,\middle|\, N=n\right]\right]\right)^2 \\
&\overset{\text{sub}}{=} E_N[N]\,\sigma_x^2 + E_N\left[\left(E\left[\sum_{k=1}^{n} X_k \,\middle|\, N=n\right]\right)^2\right] \\
&\qquad - \left(E_N\left[E\left[\sum_{k=1}^{n} X_k \,\middle|\, N=n\right]\right]\right)^2 \\
&\overset{\text{ind}}{=} E_N[N]\,\sigma_x^2 + E_N\left[\left(E\left[\sum_{k=1}^{n} X_k\right]\right)^2\right] \\
&\qquad - \left(E_N\left[E\left[\sum_{k=1}^{n} X_k\right]\right]\right)^2 \\
&\overset{\text{iid}}{=} E_N[N]\,\sigma_x^2 + \mu_x^2 \sum_{n=1}^{\infty} n^2\cdot p_N(n) - \mu_x^2\left(\sum_{n=1}^{\infty} n\cdot p_N(n)\right)^2 \\
&= E_N[N]\,\sigma_x^2 + \mu_x^2\cdot E_N[N^2] - \mu_x^2\cdot E_N^2[N] \\
&= E_N[N]\,\sigma_x^2 + \mu_x^2\left(E_N[N^2] - E_N^2[N]\right) \\
&= E_N[N]\,\sigma_x^2 + \mu_x^2\, V_N[N].
\end{aligned}
$$

QED

Corr: $N \sim \text{Poisson}(\lambda)$ ← e.g. count # cars enter freeway in one minute (e.g. suppose average = 10) ← λ.

$$E\left[\sum_{k=1}^{N} X_k\right] = \lambda \cdot \mu_X$$

$$V\left[\sum_{k=1}^{N} X_k\right] = \lambda\left(\sigma_X^2 + \mu_X^2\right) = \lambda \cdot E[X^2]$$

↑ $V[X] = E[X]$

Similarly:

Thm: ("Doubly Random Sample Mean") If

- iid $X_1, \dots, X_n$ for $\forall n$ and $\sigma_X^2 < \infty$ $(\therefore \mu_X < \infty)$
- positive $\mathbb{Z}^+$-valued r.v. N
- N is independent of $X_1, \dots, X_n$ for $\forall n$:

(1) $E[\bar{X}_N] = \mu_X$

(2) $V[\bar{X}_N] = \sigma_X^2 \cdot E_N\left[\frac{1}{N}\right]$

Prf: (at home)